9.4: Sequences and Series

Sequence: An ordered progression of numbers.

Finite - has an end

Example: 2, 4, 6, 8, ... 28

Infinite - does not end

Example: 2, 4, 5, 8, ...

Arithmetic Sequence:

A sequence where each term can be obtained by adding a **common difference** (d) to the previous term.

Geometric Sequence:

A sequence where each term can be obtained by multiplying a **common ratio** (r) to the previous term.

Notation:

a_n: The n^{th} term of a sequence a_2 : 2nd TERM

a_{n-1}: The term before a_n

Arithmetic, Geometric, or Neither?

(+) (×)

$63, 58, 53, 48, 43, \ldots$

−5 −5 −5

$\boxed{\text{ARITH } d=-5}$

$-7, -4, -1, 2, \ldots$

+3 +3

$\boxed{\text{ARITH } d=+3}$

$81, 27, 9, 3, 1, \frac{1}{3}, \ldots$

$\times\frac{1}{3}$

$\boxed{\text{GEO } r=\frac{1}{3}}$

$\frac{4}{3}, \frac{5}{3}, \cancel{2}\frac{6}{3}, \frac{7}{3}, \frac{8}{3}, \cancel{3}\frac{9}{3}$

$+\frac{1}{3}$

$\boxed{\text{ARITH } d=\frac{1}{3}}$

$2, 4, 8, 16, 32$

$\boxed{\text{GEO } r=2}$

$1, 4, 9, 16, 25, \ldots$

$\boxed{\text{NEITHER}}$

$1^2\ 2^2\ 3^2\ 4^2\ 5^2$

Defining Sequences

Recursive: A formula that defines each term of a sequence using the term(s) that come prior.

Example: $b_1 = 3 \quad b_n = b_{n-1} + 5$

Arithmetic Formula: $a_n = a_{n-1} + d$

Geometric Formula: $a_n = a_{n-1}(r)$

Explicit: A formula that defines each term of a sequence.

Example: $b_n = -2 + 5n$ $\quad b_{100} = -2 + 5(100)$

Arithmetic Formula: $a_n = a_1 + (n-1)d$

Geometric Formula: $a_n = a_1(r)^{n-1}$

Example

$-6, -2, 2, 6, 10, \ldots$

1. Find the common difference.

   $d = 4$

2. Find the 10th term.

   $a_{10} = -10 + 4(10)$
   $= -10 + 40 = \boxed{30}$

3. Find a recursive rule for the nth term.

   $a_n = a_{n-1} + 4 \quad ; \quad a_1 = -6$

4. Find an explicit rule for the nth term.

   $a_n = -6 + (n-1)(4)$

   $a_n = -6 + 4n - 4$

   $\boxed{a_n = -10 + 4n}$

Example

$3, 6, 12, 24, 48, \ldots$
x2 x2

1. Find the common ratio.

   $r = 2$

2. Find the 10th term.

   $a_{10} = 3(2)^9 = \boxed{1536}$

3. Find a recursive rule for the nth term.

   $a_n = a_{n-1}(2) \quad ; \quad a_1 = 3$

4. Find an explicit rule for the nth term.

   $a_n = 3(2)^{n-1}$

   $a_n = 3(2)^n(2)^{-1}$

   $a_n = 3(2)^n \cdot \frac{1}{2}$

   $a_n = \frac{3}{2}(2)^n$

$x^{5+2} = x^7$

$x^7 = x^5 x^2$

Example

$a_2 = 3 \qquad a_5 = 24$

$a_n = a_1 + (n-1)d$

$a_5 = a_2 + 3d$

$24 = 3 + 3d$

$21 = 3d$

$7 = d$

Given the sequence is arithmetic, find:

1. Common difference

   $d = 7$

2. First term

   $3 - 7 = \boxed{-4 = a_1}$

3. Recursive formula

   $a_n = a_{n-1} + 7 \;;\; a_1 = -4$

4. Explicit formula

   $a_n = -4 + (n-1)(7)$

Given the sequence is geometric, find:

1. Common ratio

   $r = 2$

2. First term

   $a_1 = \frac{3}{2}$

3. Recursive formula

   $a_n = a_{n-1}(2) \qquad a_1 = \frac{3}{2}$

4. Explicit formula

   $a_n = \frac{3}{2}(2)^{n-1}$

Series: The sum of the terms of a sequence.

Summation Notation: The sum of the terms of the sequence $a_1, a_2, a_3, a_4, \ldots, a_n$ is denoted $\sum_{k=1}^{n} a_k$

$\sum_{k=1}^{5} 2+3k = \underbrace{5+8+11+14+17}$

EXPLICIT FORMULA

Sum of a Finite Arithmetic Sequence:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n = n\left(\frac{a_1 + a_n}{2}\right)$$

$\frac{n}{2}(a_1 + a_n)$

Sum of a Finite Geometric Sequence:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

Sum of an Infinite Geometric Series: $\sum_{k=1}^{\infty} a_k = \frac{a_1}{1-r}$

$|r| < 1$

Infinite series - summation of an infinite sequence

Converges: A limit on the series exists (|r| is less than 1)

Diverges: A limit on the series does not exist (|r| is greater than 1)

Example

$$a_1 = 8 \qquad a_n = 24 \qquad d = 2$$

1. Write an explicit equation.

$$a_n = 8 + (n-1)(2)$$
$$= 8 + 2n - 2$$
$$\boxed{a_n = 6 + 2n}$$

2. Write the series in sigma notation.

$$\sum_{k=1}^{9} 6+2n$$

$$24 = \cancel{6} + 2n$$
$$-6 \quad \cancel{-6}$$
$$18 = 2n$$
$$9 = n$$

$$24 = a_9$$

3. Find the sum of the finite series.

$$S_9 = 9\left(\frac{8+24}{2}\right) = \boxed{144}$$

Example

$$a_1 = 4 \qquad r = -\frac{1}{3} \qquad n = 8$$

1. Write an explicit equation.

$$a_n = 4\left(\frac{-1}{3}\right)^{n-1}$$

2. Write the series in sigma notation.

$$\sum_{n=1}^{8} 4\left(-\frac{1}{3}\right)^{n-1}$$

3. Find the sum of the finite series.

$$S_8 = 4\left(\frac{1-(-1/3)^8}{1-(-1/3)}\right) = 2.9995 \approx \boxed{3}$$

Example

Will this series converge or diverge? If it converges, find the sum.

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$$

$r = -\frac{1}{2}$ CONVERGES

$$S_\infty = \frac{1}{1-(-\frac{1}{2})} = \frac{1}{1+\frac{1}{2}} = \boxed{\frac{2}{3}}$$

$$10+20+30+40+\ldots$$

$d = 10$ DIVERGES

$$.1+.01+.001+.0001+\ldots$$

$r = .1$ CONVERGES

$$S_\infty = \frac{.1}{1-.1} = \boxed{\frac{1}{9}}$$