Precalculus

When you hear the word limit, what do you think about?

Day 4: Intro to Limits

A limit is a value that is getting approached.

For example:

- The limit of the amount of teeth in a baby's moth is... 32
- The limit to the amount of shoes you can own is... infinity!
- The limit to the amount of states that the President has visited is... 50
- The limit to how many pictures Miss Wright can take of her dog is... infinity!

Now let's try some more mathematical examples. Find the limit of the following:

- $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \frac{1}{100000}, \ldots \rightarrow 0$
- $\frac{3}{2}, \frac{7}{4}, \frac{11}{6}, \frac{15}{8}, \frac{19}{10}, \ldots \rightarrow 2$
- $1, 1, 2, 3, 5, 8, 13, 21, \ldots \rightarrow$ infinity

Notation: $\lim_{x \to a} f(x) = L$ "The limit of $f(x)$ as x approaches a is L."

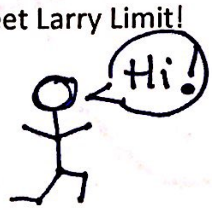
Examples: Fill in the missing information.

- $\lim_{x \to \infty} \left(\frac{1}{x^2}\right) = 0$ "The limit of $\frac{1}{x^2}$ as x approaches ∞ is 0."
- $\lim_{n \to -7} (3n) = -21$ "The limit of $3n$ as n approaches -7 is -21."
- $\lim_{x \to -9} \left(\frac{x}{18}\right) = -\frac{1}{2}$ "The limit of $x/18$ as x approaches -9 is $-\frac{1}{2}$."
- $\lim_{x \to f} f(r) = w$ "The limit of $f(r)$ as x approaches f is w."

Finite Limits:

We are going to be finding where y is approaching as x approaches a value. Every answer you give will be a y-value.

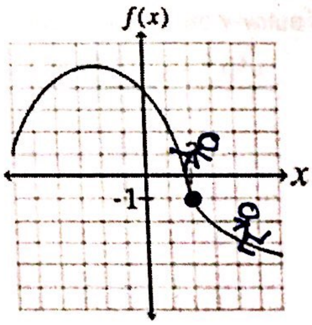
First, meet Larry Limit!

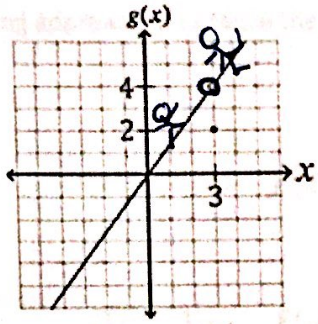


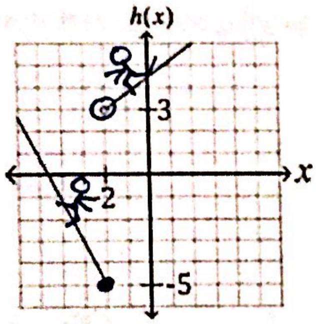
$f(2) = -1$ $\quad$ $g(3) = 2$

Most of the time, finding the limit is not very different than evaluating the function. But sometimes this is not the case, if there are holes or jumps in the graph.

$\lim_{x\to 2} f(x) = -1$ $\quad$ $\lim_{x\to 3} g(x) = 4$ $\quad$ $\lim_{x\to -2} h(x) = ?$

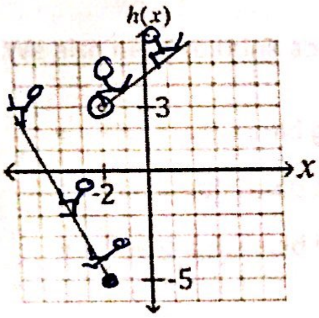






There are some cases where approaching from the right takes you somewhere different than when you are approaching from the left. First, we need notation to establish which direction we are approaching from:

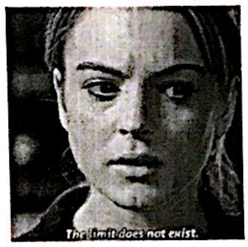
$\lim_{x\to -2} h(x) =$



Approaching $x = -2$ from the <u>left</u>: -2^{-} $\qquad$ $\lim_{x\to -2^{-}} h(x) = -5$

Approaching $x = -2$ from the <u>right</u>: -2^{+} $\qquad$ $\lim_{x\to -2^{+}} h(x) = 3$

Whenever the limit from the left and the limit from the right **do not approach the same value**, then the limit <u>DOES NOT EXIST!!</u>



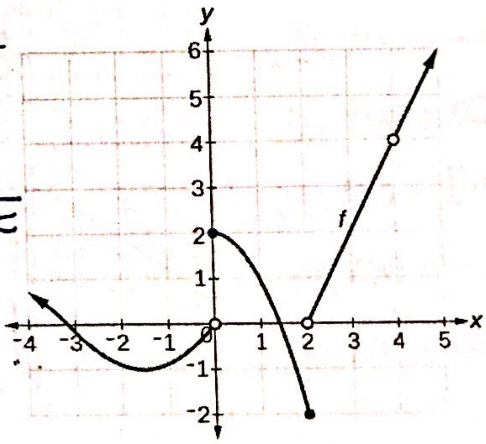
Now you know why! ☺

Practice Problems:

a. $\lim_{x\to 0^-} f(x) = 0$ $\quad$ b. $\lim_{x\to 0^+} f(x) = 2$ $\quad$ c. $\lim_{x\to 0} f(x) =$ DNE

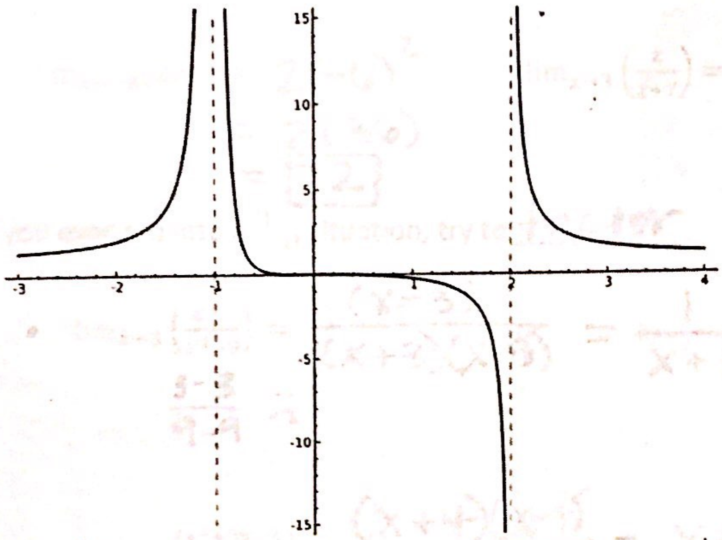
d. $\lim_{x\to 2^-} f(x) = -2$ $\quad$ e. $\lim_{x\to 2^+} f(x) = 0$ $\quad$ f. $\lim_{x\to 2} f(x) =$ DNE

g. $\lim_{x\to 4^-} f(x) = 4$ $\quad$ h. $\lim_{x\to 4^+} f(x) = 4$ $\quad$ i. $\lim_{x\to 4} f(x) = 4$



Infinite Limits:

Sometimes as we approach an x-value, there is no y-value being approached because the graph does not stop going up or down. This means that the limit is ∞ or $-\infty$.



(a) $\lim_{x \to -1^-} f(x) = \infty$

(b) $\lim_{x \to -1^+} f(x) = \infty$

(c) $\lim_{x \to 2^-} f(x) = -\infty$

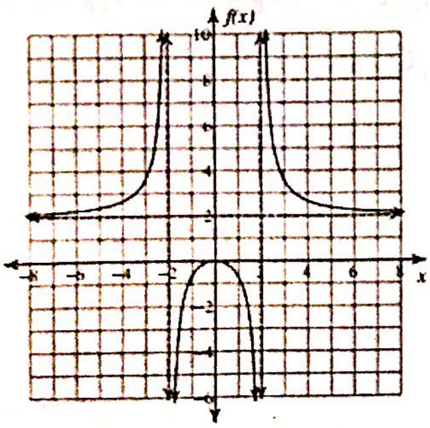
(d) $\lim_{x \to 2^+} f(x) = \infty$

We also need to think about what happens when *x* approaches positive or negative infinity:

right → left ←

- The graph could go up (infinity)
- The graph could go down (negative infinity)
- The graph could level off (y-value)

Practice Problems:



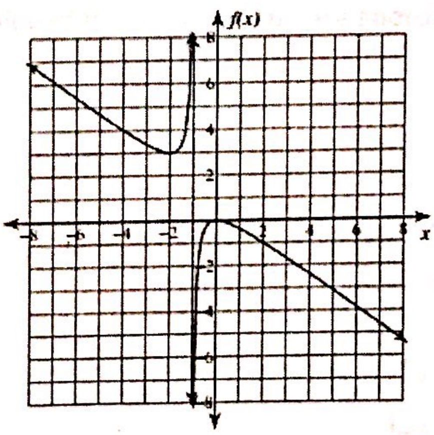
$\lim_{x \to 2^+} f(x) = \infty$

$\lim_{x \to 2^-} f(x) = -\infty$

$\lim_{x \to 2} f(x) =$ DNE

$\lim_{x \to +\infty} f(x) = 2$

$\lim_{x \to -\infty} f(x) = 2$



$\lim_{x \to -1^+} f(x) = -\infty$

$\lim_{x \to -1^-} f(x) = \infty$

$\lim_{x \to -1} f(x) =$ DNE

$\lim_{x \to +\infty} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = \infty$

Finding Limits Algebraically:

Our first step to finding limits will be to try to find the function value (substitute).

Examples:

$\lim_{x\to -6}(2x^2) = 2(-6)^2 = 2(36) = \boxed{72}$

$\lim_{x\to 3}\left(\frac{x}{x+7}\right) = \frac{3}{3+7} = \boxed{\frac{3}{10}}$

$\lim_{x\to 3}(-10) = \boxed{-10}$

If you ever run into a $^0/_0$ situation, try to factor.

- $\lim_{x\to 3}\left(\frac{x-3}{x^2-9}\right) = \frac{\cancel{(x-3)}}{(x+3)\cancel{(x-3)}} = \frac{1}{x+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}$

  $\frac{3-3}{9-9}$ ☹

- $\lim_{x\to 1}\left(\frac{x^2+3x-4}{x-1}\right) = \frac{(x+4)\cancel{(x-1)}}{\cancel{(x-1)}} = x+4 = 1+4 = \boxed{5}$

- $\lim_{x\to 7}\left(\frac{x^2-10x+21}{x^2-49}\right) = \frac{\cancel{(x-7)}(x-3)}{\cancel{(x-7)}(x+7)} = \frac{x-3}{x+7} = \frac{7-3}{7+7} = \frac{4}{14} = \boxed{\frac{2}{7}}$

If you ever run into a $^\#/_0$ situation, try test points on the side of the direction you are approaching from to determine if it will approach ∞ or $-\infty$.

$\lim_{x\to 0^+}\left(\frac{5}{x^2}\right) = \boxed{\infty}$

try 1 → $\frac{5}{(1)^2} = 5$

$\lim_{x\to 0^+}\left(\frac{x+7}{x}\right) = \boxed{\infty}$

$\frac{(1)+7}{(1)} = 8$

$\lim_{x\to 2^+}\left(\frac{6x}{4-x^2}\right) = \boxed{-\infty}$

$\frac{6(3)}{4-(3)^2} = \frac{18}{-5}$

$\lim_{x\to 0^-}\left(\frac{5}{x^2}\right) = \boxed{\infty}$

try −1 → $\frac{5}{(-1)^2} = 5$

$\lim_{x\to 0^-}\left(\frac{x+7}{x}\right) = \boxed{-\infty}$

$\frac{(-1)+7}{(-1)} = \frac{6}{-1} = -6$

$\lim_{x\to 2^-}\left(\frac{6x}{4-x^2}\right) = \boxed{\infty}$

$\frac{6(1)}{4-(1)^2} = \frac{6}{3}$

What about if ***x* is approaching infinity?** Infinity is not a number…it is the concept of never ending. It's BIG.

$\infty + 3 =$ BIG $\boxed{\infty}$

$\infty - 3 =$ still BIG $\boxed{\infty}$

$-3\infty =$ − BIG $\boxed{-\infty}$

$\infty/3 =$ still BIG $\boxed{\infty}$

However, **negatives DO MATTER!**

$\lim_{x\to\infty}(2x^2) =$ BIG $= \boxed{\infty}$

$\lim_{x\to\infty}\left(\frac{4}{x}\right) = \frac{4}{\text{BIG}} = \boxed{0}$

$\lim_{x\to\infty}\left(\frac{x}{4}\right) = \frac{\text{BIG}}{4} = \boxed{\infty}$

$\lim_{x\to-\infty}(2x^2) = \boxed{\infty}$

$2(-\infty)^2$ ↑ positive

$\lim_{x\to-\infty}\left(\frac{4}{x}\right) = \frac{4}{-\text{BIG}} = \boxed{0}$

$\lim_{x\to-\infty}\left(\frac{x}{4}\right) = \frac{-\text{BIG}}{4} = \boxed{-\infty}$

Rules for when rational functions approach infinity:

1. If the degree on the top is bigger, it goes to infinity.
2. If the degree on the bottom is bigger, it goes to zero.
3. If the degrees are the same, divide the leading coefficients!

Sound familiar? ☺ Horizontal & slant asymptotes!!

Examples:

- $\lim_{x\to\infty}\left(\frac{x^2+3x-4}{x+1}\right) = \frac{\text{BIGGER}}{\text{BIG}} = \boxed{\infty}$
- $\lim_{x\to\infty}\left(\frac{3x+4}{2x+1}\right) = \frac{3\cdot\cancel{\text{BIG}}}{2\cdot\cancel{\text{BIG}}} = \boxed{\frac{3}{2}}$
- $\lim_{x\to\infty}\left(\frac{6x-5}{x^3-1}\right) = \frac{\text{BIG}}{\text{BIGGER}} = \boxed{0}$

CHALLENGE: $\lim_{x\to\infty}\left(\frac{\sin x}{x}\right) = \frac{\pm 1}{\text{BIG}} = \boxed{0}$ $\lim_{x\to\infty}(\sin x) = \boxed{\text{DNE}}$

+1? −1?